ON POLYGONS OF RATIONAL SIDES AND RATIONAL VERTICES

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Abstract. In this note, we study polygons in the 2-D plane with rational vertices and rational side lengths and show that a regular rational polygon cannot be inscribed in the unit circle.

1. Introduction

Definition 1.1. An n-tuple of points \((P_1, P_2, P_3, \ldots, P_n)\) form a rational polygon if \(\forall i \leq n (P_i = (x_i, y_i) \in \mathbb{Q} \times \mathbb{Q})\) and \(\forall i < n (\|P_i - P_{i+1}\| \in \mathbb{Q})\)

Definition 1.2. A rational distance set \(S\) is a set of elements of the form \(P = (x, y) \in \mathbb{R}^2\), such that \(\forall P_i \in S \forall P_j \in S (\|P_i - P_j\| \in \mathbb{Q})\)

A rational distance set defined in Definition 1.2 is different from Definition 1.1 in the sense that a rational distance set could be infinite while a rational polygon is necessarily finite. A rational distance set does not necessarily have rational coordinates for its points.

Example 1.3. (Rational Distance Set) The set \(S := \{(1, 0), \left(1, \frac{-3}{4}\right), (1, \frac{4}{3})\}\) is a rational distance set. Moreover, \(S + \sqrt{2}\) (where the addition is point-wise addition) is also a rational distance set but of irrational point coordinates.

Proposition 1.4. A finite rational distance set of \(n\) rational points with no three of them collinear are the vertices of a rational \(n\)-gon.

Moreover, by simple polygonization, one can construct a simple polygon from a rational distance set.

2. Rational Polygons on the Unit Circle

The unit circle \(C\) can be parametrized by the parametrization \(\rho : \mathbb{R} \rightarrow C\) given by:

\[
\rho(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^4}\right)
\]

In particular, the set of rational points on the unit circle is \(\rho(\mathbb{Q})\). In other words, the set of rational points on the unit circle can be parametrized by the restriction of \(\rho\) to \(\mathbb{Q}\) \((\rho|_{\mathbb{Q}} : \mathbb{Q} \rightarrow C)\) as follows:

\[
\rho\left(\frac{n}{m}\right) = \left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}\right)
\]

\((C, \oplus)\) is an abelian group under “angle addition” [1]. Given two points \((x_1, y_1)\) and \((x_2, y_2)\) on \(C\), the group addition \(\oplus\) is defined by:

\[
(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)
\]
The rational points on $C$ form a subgroup of $(C, \oplus)$ denoted by $C(\mathbb{Q})$ [1].

[3] shows that given any circle, there exists a dense rational distance set of rational points that lie on that circle. They first construct a rational distance set of rational points on a line and then map it via a Möbius transformation on a given circle. For the unit circle, consider the set of rational points on the line $L : \text{Re}(z) = \frac{1}{2}$,

\begin{equation}
S := \left\{ \frac{1}{2} \left( 1 + i \frac{s^2 - 1}{2s} \right) \bigg| s \in \mathbb{Q} \right\}
\end{equation}

One can easily verify that $S$ is a rational distance set on $L$. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the Möbius transformation given by:

\begin{equation}
f(z) = -\frac{z - 1}{z}
\end{equation}

$f(S)$ is indeed a rational distance set and the proof is outlined in [3]. Now, we collect this discussion in the following theorem which is a special case of Theorem 2.2. in [3].

**Theorem 2.1.** ([3]) There exists a dense rational distance set on the unit circle $C$ consisting of rational points.

**Proposition 2.2.** For $n \geq 3$, there exists a rational $n$-gon inscribed in the unit circle.

**Proof.** By Theorem 2.1 and Proposition 1.4, and given that no three points are collinear on a circle, Proposition 2.2 follows directly by the map $f(A)$ where $A \subset S$ given by restricting $s$ in equation 2.4 to and only to $\{1, \ldots, n\} \subset \mathbb{Q}$. □

Natural questions that arise after establishing the fact that there exist infinitely many rational polygons on the unit circle are usually related to the existence of special rational polygons. We show that a regular rational polygon cannot be inscribed in the unit circle.

**Theorem 2.3.** (Niven [2]) If $\theta$ is in the interval $[0, 90]$ and $\sin(\frac{\pi \theta}{180}) \in \mathbb{Q}$, then $\theta \in \{0, 30, 90\}$

**Lemma 2.4.** If $n \geq 4$, a regular rational $n$-gon cannot be inscribed in the unit circle if $(0, 1)$ is a vertex.

**Proof.** Let $(P_1, \ldots, P_n)$ be a regular rational $n$-gon where $P_1 = (0, 1)$ and $n \geq 4$. By the rational parametrization, $\exists t (\rho(t) = P_2)$. Since the $y$-coordinate of $P_2$ is rational, by Niven’s theorem, one must have $2 \arctan(t)$ in degrees be either $30^\circ$ (a regular dodecagon) or $90^\circ$ (a square). The side length in the case of a square is $\sqrt{2} \notin \mathbb{Q}$ while in the case of a dodecagon, the $x$-coordinate is irrational ($\frac{\sqrt{3}}{2}$). □

**Theorem 2.5.** A regular rational polygon cannot be inscribed in the unit circle.

**Proof.** An equilateral triangle inscribed in the unit circle has a side length of $\sqrt{3} \notin \mathbb{Q}$. For $n \geq 4$, assume for the sake of contradiction that $P = (P_1, \ldots, P_n)$ is a regular rational polygon. Consider the polygon $P' = (P'_1, \ldots, P'_n)$ where $P'_1 = (1, 0)$ and for $k > 1$, $P'_k := \bigoplus_{i=1}^{k-1}(P_2 \oplus (-P_1))$. Clearly, by the group structure of $C(\mathbb{Q})$, the vertices of $P'$ are in $C(\mathbb{Q})$ and since this is a rotational transformation in disguise, the side lengths are preserved. The existence of such $P'$ contradicts Lemma 2.4 □
References

